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Speciality of Malcev superalgebras on one odd generator

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Abstract

It is proved that every Malcev superalgebra generated by an odd element is special, that is, isomorphic to a subsuperalgebra of the commutator Malcev superalgebra A^- for a certain alternative superalgebra A. As a corollary, it is shown that the kernel of the natural homomorphism of the free Malcev algebra Malc$[X]$ of countable rank into the commutator Malcev algebra Alt$[X]^-$ of the corresponding free alternative algebra Alt$[X]$, does not contain skew-symmetric multilinear elements. In other words, there are no skew-symmetric Malcev s-identities. Another corollary is speciality of the Malcev Grassmann algebra.

1. Introduction

Malcev algebras were introduced in 1955 (under the name Moufang–Lie algebras) by A.I. Malcev [4] as tangent algebras of analytic Moufang loops. They generalize Lie algebras and are related to alternative algebras in the same way as Lie algebras are related to

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associative algebras: if A is an alternative algebra, then the algebra A^- obtained from an algebra A by replacing the product $x \cdot y$ with the commutator $[x, y] = x \cdot y - y \cdot x$ is a Malcev algebra. Although the theory of Malcev algebras is quite well developed (see [3]), it is still an open problem whether the analog of the Poincaré–Birkhoff–Witt theorem is true for them; that is, whether any Malcev algebra is isomorphic to a subalgebra of an algebra A^- for a suitable alternative algebra A [2,7,8]. A Malcev algebra M that admits such a representation is called *special*.

The speciality problem is particularly interesting for free algebras. In this case it is equivalent to the question whether the kernel of the natural homomorphism of the free Malcev algebra of countable rank into the commutator Malcev algebra A^- of the corresponding free alternative algebra A is nonzero. Since no effective bases are known either for free Malcev algebras or for free alternative algebras (see [1, Problem 1.160]), the question is far from being evident.

Hypothetical nonzero elements from the above-mentioned kernel are called, by analogy with the Jordan case, *Malcev s-identities*; they vanish in every special Malcev algebra. It seems natural to ask first whether there could exist Malcev s-identities of special type, for instance, skew-symmetric s-identities. In this case, due to [6,9,11], the problem is reduced to free Malcev and alternative superalgebras on one odd generator, which are easier to deal with.

In [7], the free Malcev superalgebra $\mathcal{M}$ on one odd generator was investigated. In particular, a base of $\mathcal{M}$ was constructed and it was proved that $\mathcal{M}$ is central-by-metabelian, that is, satisfies the equality $(\mathcal{M}^2\mathcal{M}^2)\mathcal{M} = 0$. In [10] (see also [13]), a base of the Malcev Poisson superalgebra related with $\mathcal{M}$ according to [8] was constructed. As a corollary, a prebase, that is, a set of elements that spans the free alternative superalgebra $\mathcal{A}$ on one odd generator, was obtained. However, it remained an open question whether this set forms a base of $\mathcal{A}$. Accordingly, the problem of speciality of the superalgebra $\mathcal{M}$ remained open as well.

Here we solve this problem in the affirmative. More exactly, we construct a base of the free central-by-metabelian alternative superalgebra on one odd generator $\mathcal{E} = \mathcal{A}/I$, where $I = (\mathcal{A}^2 \cdot \mathcal{A}^2) \cdot \mathcal{A} + \mathcal{A} \cdot (\mathcal{A}^2 \cdot \mathcal{A}^2)$. The natural homomorphism of $\mathcal{M}$ into the commutator superalgebra $\mathcal{E}^-$ turns out to be injective, and hence the superalgebra $\mathcal{M}$ is special. As a corollary, we conclude that there are no nontrivial skew-symmetric Malcev s-identities. Another corollary of our result is the speciality of the Malcev Grassmann algebra which was introduced in [9]. Finally, we consider the homomorphic images of $\mathcal{M}$ and prove that they are special as well.

Below all the (super)algebras will be over a field F of characteristic different from 2 and 3.

2. Relations in the universal alternative envelope

Recall that an anticommutative algebra M is called a *Malcev algebra* if it satisfies the identity

$$J(x, y, z)x = J(x, y, xz), \tag{1}$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of x, y and z.

An algebra A is called *alternative* if it satisfies the identities

$$(x, x, y) = 0 \quad (\textit{left alternativity}),$$
$$(x, y, y) = 0 \quad (\textit{right alternativity}),$$

where $(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z)$ is the associator of x, y and z. Here and below we denote multiplication in alternative and in Malcev (super)algebras by dot $x \cdot y$ and by juxtaposition xy, respectively.

As we mentioned in the introduction, for every alternative algebra A the commutator algebra A^- with the product $[x, y] = x \cdot y - y \cdot x$ is a Malcev algebra. By standard arguments, one can prove that for any Malcev algebra M there exist an alternative algebra $U(M)$ and a Malcev algebra homomorphism $\tau : M \to U(M)^-$ with the following universal property: for any homomorphism of Malcev algebras $\rho : M \to A^-$, where A is alternative, there exists a unique homomorphism of alternative algebras $\hat{\rho} : U(M) \to A$ such that $\rho = \hat{\rho} \circ \tau$. The algebra $U(M)$ is called the *universal alternative envelope of* M. Note that M is special if and only if $\ker \tau = 0$.

We will consider Malcev and alternative superalgebras. Recall that, in general, a *superalgebra* means a $\mathbb{Z}_2$-graded algebra, that is an algebra A which may be written as a direct sum of subspaces $A = A_0 + A_1$ subject to the relation $A_i \cdot A_j \subseteq A_{i+j \ (\mathrm{mod}\ 2)}$. The subspaces A_0 and A_1 are called the *even* and the *odd* parts of the superalgebra A; and so are called the elements from A_0 and from A_1, respectively. Below all the elements are assumed to be homogeneous, that is, either even or odd, and for an element $x \in A_i$, $i \in \{0, 1\}$, the symbol $\bar{x} = i$ means its parity.

If the characteristic of the ground field is different from 2, identity (1) is equivalent to the following multilinear identity

$$\big((xy)z\big)t + \big((yz)t\big)x + \big((zt)x\big)y + \big((tx)y\big)z = (xz)(yt),$$

which was proved in [5, Proposition 2.21]. Now, a superalgebra $M = M_0 + M_1$ is called a *Malcev superalgebra* if it satisfies the following super-identities:

$$xy = -(-1)^{\bar{x}\bar{y}} yx \quad (\textit{super-anticommutativity}),$$
$$\big((xy)z\big)t - x\big((yz)t\big) - (-1)^{\bar{y}(\bar{z}+\bar{t})}\big(x(zt)\big)y - (-1)^{\bar{t}(\bar{y}+\bar{z})}\big((xt)y\big)z$$
$$= (-1)^{\bar{y}\bar{z}}(xz)(yt) \quad (\textit{super-Malcev identity}).$$

Note that the super-identities are obtained from anticommutativity and Sagle's multilinear identity, respectively, by the so-called "superization rule" (or "Kaplansky's principle") that whenever two odd variables are transposed a negative sign is introduced.

Similarly, a superalgebra $A = A_0 + A_1$ is called an *alternative superalgebra* if it satisfies

$$(x, y, z) + (-1)^{\bar{x}\bar{y}}(y, x, z) = 0 \quad (\textit{left super-alternativity}),$$
$$(x, y, z) + (-1)^{\bar{y}\bar{z}}(x, z, y) = 0 \quad (\textit{right super-alternativity}).$$

The superalgebra A^- is obtained from a superalgebra A by replacing the product $x \cdot y$ with the super-commutator $[x, y]_s = x \cdot y - (-1)^{\bar{x}\bar{y}} y \cdot x$.

Denote by $\mathcal{M} = \mathrm{Malc}[x]$ the free Malcev superalgebra and by $\mathcal{A} = \mathrm{Alt}[x]$ the free alternative superalgebra on one odd generator x. It is clear that $\mathcal{A} \cong U(\mathcal{M})$, with the universal homomorphism $\tau : \mathcal{M} \to \mathcal{A}^-$ defined by $\tau(x) = x$. Thus we can identify $\mathcal{A}$ and $U(\mathcal{M})$.

Observe that for homogeneous elements $u, v, w \in \mathcal{M}$ we have

$$\begin{gathered}\bigl[\tau(u), \tau(v)\bigr]_s = \tau(uv),\\ \bigl(\tau(u), \tau(v), \tau(w)\bigr) = \tfrac{1}{6}\tau\bigl(J_s(u, v, w)\bigr),\end{gathered} \tag{2}$$

where

$$J_s(u, v, w) = (uv)w + (-1)^{\bar{u}(\bar{v}+\bar{w})}(vw)u + (-1)^{\bar{w}(\bar{u}+\bar{v})}(wu)v$$

is the super-Jacobian of u, v and w.

It was proved in [7] that the superalgebra $\mathcal{M}$ has a base

$$\mathcal{B}_{\mathcal{M}} = \bigl\{x^k,\ x^{4k}x^2,\ x^{4k+1}x^2 \mid k > 0\bigr\},$$

with the nonzero products given, up to super-anticommutativity, by

$$\begin{aligned} x^i x &= x^{i+1},\\ x^i x^j &= -(-1)^{c(j)} x^{i+j-2}x^2, \quad i + j \in \{4k+2, 4k+3\},\ k > 0,\end{aligned}$$

where $c(j) = \frac{j(j-1)}{2}$. Observe that the function c satisfies the equality

$$c(i + j) = c(i) + c(j) + ij; \tag{3}$$

in particular, $c(i + 1) = c(i) + i$.

It follows from the multiplication table of $\mathcal{M}$ that the elements $x^{4k}x^2$, $x^{4k+1}x^2$ annihilate $\mathcal{M}$. Therefore, the super-Jacobians J_s can be nonzero on the base $\mathcal{B}_{\mathcal{M}}$ only for the arguments x^i, where at least one of them equals x.

As in [10], we obtain for $i, j > 1$

$$\begin{aligned} J_s\bigl(x^i, x^j, x\bigr) &= \begin{cases} 2(-1)^{c(j+1)}x^{i+j-1}x^2, & i + j + 1 \in \{4k+2, 4k+3\},\\ 0, & i + j + 1 \notin \{4k+2, 4k+3\},\end{cases}\\ J_s\bigl(x^i, x, x\bigr) &= 2x^{i+2} - x^i x^2,\\ J_s(x, x, x) &= 3x^3.\end{aligned}$$

Denote $\tau(x^i) = x^{[i]}$, $\tau(x^i x^2) = z^{[i]}$, $i > 1$. Then in the superalgebra $\mathcal{A}$ we have

$$\begin{aligned} x^{[2]} &= 2x \cdot x,\\ x^{[3]} &= 2(x \cdot x) \cdot x - 2x \cdot (x \cdot x) = 2(x, x, x),\end{aligned}$$

and for $i, j > 1$ we obtain in view of (2)

$$\big(x^{[i]}, x, x\big) = \tfrac{1}{3}x^{[i+2]} - \tfrac{1}{6}z^{[i]}, \tag{4}$$

$$\big(x^{[i]}, x^{[j]}, x\big) = \tfrac{1}{3}(-1)^{c(j+1)}z^{[i+j-1]}. \tag{5}$$

Let $a \circ b = a \cdot b + b \cdot a$ stands for the Jordan product of (nongraded) elements a and b, and $a \circ_s b = a \cdot b + (-1)^{\bar{a}\bar{b}} b \cdot a$ means the super-Jordan product of homogeneous elements $a, b \in \mathcal{A}$. For any homogeneous $v \in \mathcal{M}$ and $i > 1$ we have

$$\big[z^{[i]}, \tau(v)\big]_s = \big[\tau\big(x^i x^2\big), \tau(v)\big]_s = \tau\big(\big(x^i x^2\big)v\big) = 0,$$

hence

$$z^{[i]} \circ_s \tau(v) = 2z^{[i]} \cdot \tau(v) - \big[z^{[i]}, \tau(v)\big]_s = 2z^{[i]} \cdot \tau(v). \tag{6}$$

Proposition 2.1. *In the superalgebra $\mathcal{A}$, the following relations hold for any $i, j > 2$:*

$$\begin{aligned} z^{[i]} \cdot z^{[j]} &= 0, \\ x^{[i]} \cdot z^{[j]} &= 0, \\ x^{[i]} \circ_s x^{[j]} &= (-1)^{c(j+1)}\big(u^{[i+j-3]} + \big(1 + (-1)^j\big)z^{[i+j-4]} \cdot x^{[2]}\big), \end{aligned} \tag{7}$$

where $u^{[i]} = x^{[i]} \circ_s x^{[3]}$. Furthermore, for any $k > 0$ holds

$$u^{[4k-1]} = 0, \qquad u^{[4k+2]} = -z^{[4k+1]} \cdot x^{[2]}. \tag{8}$$

Proof. Recall that every alternative algebra satisfies the identity (see [12, Lemma 7.9, p. 145 of the English edition])

$$(u, v, w) \circ [v, w] = 0.$$

Linearizing and superizing this identity, we get the following super-identity in $\mathcal{A}$:

$$\begin{aligned} A_s(w, u_1, v_1, u_2, v_2) &= (w, u_1, v_1) \circ_s [u_2, v_2]_s \\ &\quad + (-1)^{\bar{u}_1\bar{v}_1 + \bar{u}_2(\bar{u}_1 + \bar{v}_1)}(w, u_2, v_1) \circ_s [u_1, v_2]_s \\ &\quad + (-1)^{\bar{u}_2\bar{v}_2 + \bar{v}_1(\bar{u}_2 + \bar{v}_2)}(w, u_1, v_2) \circ_s [u_2, v_1]_s \\ &\quad + (-1)^{(\bar{u}_1 + \bar{v}_1)(\bar{u}_2 + \bar{v}_2)}(w, u_2, v_2) \circ_s [u_1, v_1]_s \\ &= 0. \end{aligned}$$

By the same arguments as in [10], considering the equalities

$$A_s\big(x, x^{[2]}, x^{[k]}, x^{[2]}, x^{[n]}\big) = 0, \qquad A_s\big(x^{[k]}, x, x, x^{[n]}, x^{[m]}\big) = 0,$$

for any $i, j > 2$ we obtain the relations

$$z^{[i]} \circ_s z^{[j]} = 0, \qquad x^{[i]} \circ_s z^{[j]} = 0,$$

which in view of (6) imply the first two equalities of the proposition.

Consider now the expressions $A_s(x, x, x, x^{[i]}, x^{[j]}) = 0$ for $i, j > 1$. We have

$$\begin{aligned} A_s\bigl(x, x, x, x^{[i]}, x^{[j]}\bigr) &= (x, x, x) \circ_s \bigl[x^{[i]}, x^{[j]}\bigr]_s - (-1)^{i+j}\bigl(x^{[i]}, x, x\bigr) \circ_s \bigl[x^{[j]}, x\bigr]_s \\ &\quad + (-1)^{ij+j+i}\bigl(x^{[j]}, x, x\bigr) \circ_s \bigl[x^{[i]}, x\bigr]_s \\ &\quad + (-1)^{i+j}\bigl(x^{[i]}, x^{[j]}, x\bigr) \circ_s [x, x]_s \\ &= 0, \end{aligned}$$

which can be rewritten as

$$x^{[i+1]} \circ_s x^{[j+2]} = (-1)^j x^{[i+2]} \circ_s x^{[j+1]} - (-1)^{c(j)} z^{[i+j-1]} \circ_s x^{[2]}. \tag{9}$$

If $j > 2$, we may apply again this formula to the first term on the right side to get

$$x^{[i+1]} \circ_s x^{[j+2]} = -x^{[i+3]} \circ_s x^{[j]}.$$

Thus, for $i > 2$, $k > 0$ we have

$$\begin{aligned} x^{[i]} \circ_s x^{[4k+3]} &= -x^{[i+2]} \circ_s x^{[4k+1]} = x^{[i+4k]} \circ_s x^{[3]}, \\ x^{[i]} \circ_s x^{[4k+4]} &= -x^{[i+2]} \circ_s x^{[4k+2]} = x^{[i+4k]} \circ_s x^{[4]}, \end{aligned}$$

or for $i, j > 2$

$$x^{[i]} \circ_s x^{[j]} = \begin{cases} (-1)^{c(j+1)} x^{[i+j-3]} \circ_s x^{[3]}, & j \text{ is odd;} \\ (-1)^{c(j+1)} x^{[i+j-4]} \circ_s x^{[4]}, & j \text{ is even.} \end{cases}$$

Moreover, by (9),

$$x^{[i+j-4]} \circ_s x^{[4]} = x^{[i+j-3]} \circ_s x^{[3]} + z^{[i+j-4]} \circ_s x^{[2]},$$

which in view of (6) implies (7).

Note finally that, by super-commutativity of the product $a \circ_s b$ and by (7),

$$\begin{aligned} u^{[4k-1]} &= -x^{[3]} \circ_s x^{[4k-1]} = -u^{[4k-1]}, \\ u^{[4k+2]} &= x^{[3]} \circ_s x^{[4k+2]} = -u^{[4k+2]} - 2z^{[4k+1]} \cdot x^{[2]}, \end{aligned}$$

which yields (8). $\square$

3. A nonuniversal alternative envelope

We know from [7] that $(\mathcal{M}^2\mathcal{M}^2)\mathcal{M} = 0$. Let us try to find an alternative envelope $\mathcal{E}$ of $\mathcal{M}$ satisfying $(\mathcal{E}^2 \cdot \mathcal{E}^2) \cdot \mathcal{E} = \mathcal{E} \cdot (\mathcal{E}^2 \cdot \mathcal{E}^2) = 0$. For this purpose, consider in the universal alternative envelope $\mathcal{A}$ the subset $I = (\mathcal{A}^2 \cdot \mathcal{A}^2) \cdot \mathcal{A} + \mathcal{A} \cdot (\mathcal{A}^2 \cdot \mathcal{A}^2)$. By [12, Proposition 5.1, p. 115 of the English edition], I is an ideal of $\mathcal{A}$, and we denote by $\mathcal{E}$ the quotient superalgebra $\mathcal{A}/I$. In view of [10, Corollary 3.4], $\mathcal{E}$ is spanned by the elements

$$x^{[k]},\ x^{[k+1]} \cdot x,\ x^{[k+1]} \cdot x^{[2]},\ x^{[4k+\varepsilon]} \cdot x^{[3]},\ z^{[4k+\varepsilon]},$$

where $k > 0$, $\varepsilon \in \{0, 1\}$. Observe that

$$2x^{[4k+\varepsilon]} \cdot x^{[3]} = x^{[4k+\varepsilon]} \circ_s x^{[3]} + \left[x^{[4k+\varepsilon]}, x^{[3]}\right]_s = u^{[4k+\varepsilon]} + z^{[4k+1+\varepsilon]}.$$

Hence the elements

$$x^{[k]},\ x^{[k+1]} \cdot x,\ x^{[k+1]} \cdot x^{[2]},\ u^{[4k+\varepsilon]},\ z^{[4k+\varepsilon]},$$

where $k > 0$, $\varepsilon \in \{0, 1\}$, also span $\mathcal{E}$. Note that the elements of the last three types annihilate the whole pre-base. Let us compute the other products.

Lemma 3.1. *For $i, j > 1$ and $i + j > 4$ we have*

$$\begin{aligned} x \cdot x &= \tfrac{1}{2}x^{[2]}, \\ x \cdot x^{[i]} &= (-1)^i \left(x^{[i]} \cdot x - x^{[i+1]}\right), \\ 2x^{[i]} \cdot x^{[j]} &= (-1)^{c(j+1)} u^{[i+j-3]} - (-1)^{c(j)} z^{[i+j-2]}. \end{aligned}$$

Proof. In fact,

$$2x \cdot x = [x, x]_s = x^{[2]},$$

and for $i > 1$ we have

$$x \cdot x^{[i]} = (-1)^i x^{[i]} \cdot x + \left[x, x^{[i]}\right]_s = (-1)^i \left(x^{[i]} \cdot x - x^{[i+1]}\right).$$

To find $x^{[i]} \cdot x^{[j]}$, we write

$$2x^{[i]} \cdot x^{[j]} = x^{[i]} \circ_s x^{[j]} + \left[x^{[i]}, x^{[j]}\right]_s.$$

The second summand we compute in $\mathcal{M}$:

$$\left[x^{[i]}, x^{[j]}\right]_s = \tau\left(x^i x^j\right) = -(-1)^{c(j)} z^{[i+j-2]}.$$

For the first one, we separate the cases when $i, j > 2$ and when one of i, j equals 2. Observe that $z^{[i]} \cdot x^{[2]} = 0$ in $\mathcal{E}$, therefore for $i, j > 2$ we have by (7)

$$x^{[i]} \circ_s x^{[j]} = (-1)^{c(j+1)} u^{[i+j-3]}.$$

Adding this formula to the previous one, we get the required equality. Let now $i > 2$, $j = 2$, then we have

$$\begin{aligned} x^{[i]} \circ_s x^{[2]} &= x^{[2]} \circ_s x^{[i]} = x^{[2]} \circ_s \left[x^{[i-1]}, x\right]_s = \left[x^{[2]} \circ_s x^{[i-1]}, x\right]_s - x^{[i-1]} \circ_s \left[x^{[2]}, x\right] \\ &= -x^{[i-1]} \circ_s x^{[3]} = -u^{[i-1]}, \end{aligned}$$

which is what we need. $\square$

Notice that for $i = 3$ the last formula gives $u^{[2]} \overset{\text{def}}{=} x^{[2]} \circ_s x^{[3]} = -u^{[2]}$, which together with (8) shows that in $\mathcal{E}$ we have

$$u^{[4k-2]} = 0, \quad k > 0.$$

Corollary 3.2. *The superalgebra $\mathcal{E}$ is spanned by the elements*

$$x^{[k]},\ x^{[k+1]} \cdot x,\ x^{[2]} \cdot x^{[2]},\ u^{[4k+\varepsilon]},\ z^{[4k+\varepsilon]}, \tag{10}$$

where $k > 0$, $\varepsilon \in \{0, 1\}$.

Continue our computations. For $i > 1$ use (4) to get

$$\begin{aligned} \left(x^{[i]} \cdot x\right) \cdot x &= x^{[i]} \cdot (x \cdot x) + \left(x^{[i]}, x, x\right) = \tfrac{1}{2} x^{[i]} \cdot x^{[2]} + \tfrac{1}{3} x^{[i+2]} - \tfrac{1}{6} z^{[i]}, \\ x \cdot \left(x^{[i]} \cdot x\right) &= \left(x \cdot x^{[i]}\right) \cdot x - \left(x, x^{[i]}, x\right) \\ &= (-1)^i \left(x^{[i]} \cdot x - x^{[i+1]}\right) \cdot x + (-1)^i \left(\tfrac{1}{3} x^{[i+2]} - \tfrac{1}{6} z^{[i]}\right) \\ &= (-1)^i \left(\tfrac{2}{3} x^{[i+2]} - x^{[i+1]} \cdot x + \tfrac{1}{2} x^{[i]} \cdot x^{[2]} - \tfrac{1}{3} z^{[i]}\right). \end{aligned}$$

In view of (5), for $i, j > 1$ we have

$$\begin{aligned} x^{[i]} \cdot \left(x^{[j]} \cdot x\right) &= -\left(x^{[i]}, x^{[j]}, x\right) = -\tfrac{1}{3} (-1)^{c(j+1)} z^{[i+j-1]}, \\ \left(x^{[i]} \cdot x\right) \cdot x^{[j]} &= x^{[i]} \cdot \left(x \cdot x^{[j]}\right) + \left(x^{[i]}, x, x^{[j]}\right) \\ &= (-1)^j x^{[i]} \cdot \left(x^{[j]} x - x^{[j+1]}\right) - (-1)^j \left(x^{[i]}, x^{[j]}, x\right) \\ &= (-1)^{j+1} \left(x^{[i]} \cdot x^{[j+1]} + \tfrac{2}{3} (-1)^{c(j+1)} z^{[i+j-1]}\right). \end{aligned}$$

Finally, since right super-alternativity $(a, b, c) + (-1)^{\bar{b}\bar{c}} (a, c, b) = 0$ can be written in the form $(a \cdot b) \cdot c = a \cdot (b \circ_s c) - (-1)^{\bar{b}\bar{c}} (a \cdot c) \cdot b$, we obtain

$$\begin{aligned}\big(x^{[i]}\cdot x\big)\cdot\big(x^{[j]}\cdot x\big) &= x^{[i]}\cdot\big(x\cdot\big(x^{[j]}\cdot x\big)+(-1)^{j+1}\big(x^{[j]}\cdot x\big)\cdot x\big)\\ &= (-1)^j x^{[i]}\cdot\big(\tfrac{1}{3}x^{[j+2]}-x^{[j+1]}\cdot x-\tfrac{1}{6}z^{[j]}\big)\\ &= \tfrac{1}{3}(-1)^j\big(x^{[i]}\cdot x^{[j+2]}-(-1)^{c(j)}z^{[i+j]}\big).\end{aligned}$$

Substituting in the obtained results the expressions for the products $x^{[i]}\cdot x^{[j]}$ from Lemma 3.1, we get the following

Lemma 3.3. *For $i, j > 1$ we have*

$$\begin{aligned}x\cdot\big(x^{[2]}\cdot x\big) &= \tfrac{2}{3}x^{[4]}-x^{[3]}\cdot x+\tfrac{1}{2}x^{[2]}\cdot x^{[2]},\\ x\cdot\big(x^{[i+1]}\cdot x\big) &= (-1)^{i+1}\big(\tfrac{2}{3}x^{[i+3]}-x^{[i+2]}\cdot x-\tfrac{1}{4}u^{[i]}-\tfrac{1}{12}z^{[i+1]}\big),\\ \big(x^{[2]}\cdot x\big)\cdot x &= \tfrac{1}{3}x^{[4]}+\tfrac{1}{2}x^{[2]}\cdot x^{[2]},\\ \big(x^{[i+1]}\cdot x\big)\cdot x &= \tfrac{1}{3}x^{[i+3]}-\tfrac{1}{4}u^{[i]}+\tfrac{1}{12}z^{[i+1]},\\ x^{[i]}\cdot\big(x^{[j]}\cdot x\big) &= -\tfrac{1}{3}(-1)^{c(j+1)}z^{[i+j-1]},\\ \big(x^{[i]}\cdot x\big)\cdot x^{[j]} &= \tfrac{1}{2}(-1)^{c(j+1)}u^{[i+j-2]}-\tfrac{1}{6}(-1)^{c(j)}z^{[i+j-1]},\\ \big(x^{[i]}\cdot x\big)\cdot\big(x^{[j]}\cdot x\big) &= \tfrac{1}{6}(-1)^{c(j-2)}u^{[i+j-1]}+\tfrac{1}{6}(-1)^{c(j-1)}z^{[i+j]}.\end{aligned}$$

4. The pre-base is a base

In order to prove that the set (10) forms a base of the superalgebra $\mathcal{E}$, we will need the following example. Consider the vector space U over F with the base

$$\mathcal{B}_U=\big\{x_k,\ y_{k+1},\ u_1,\ u_{4k+\varepsilon},\ z_{4k+\varepsilon}\mid k>0,\ \varepsilon\in\{0,1\}\big\},$$

and define on it a multiplication $*$. First of all, the elements u_i and z_i annihilate the whole base. Thus we need to define products only for the elements x_i, y_j. We denote $x=x_1$, and for $i, j > 1$ set

$$\begin{aligned}x*x &= \tfrac{1}{2}x_2,\\ x_i*x &= y_i,\\ x*x_i &= (-1)^i(y_i-x_{i+1}),\\ x_i*x_j &= \tfrac{1}{2}(-1)^{c(j+1)}u_{i+j-3}+\tfrac{1}{2}(-1)^{c(j+2)}z_{i+j-2},\\ x*y_i &= (-1)^i\big(\tfrac{2}{3}x_{i+2}-y_{i+1}-\tfrac{1}{4}u_{i-1}-\tfrac{1}{12}z_i\big),\\ y_i*x &= \tfrac{1}{3}x_{i+2}-\tfrac{1}{4}u_{i-1}+\tfrac{1}{12}z_i,\\ x_i*y_j &= -\tfrac{1}{3}(-1)^{c(j+1)}z_{i+j-1},\end{aligned}$$

$$y_i * x_j = \tfrac{1}{2}(-1)^{c(j+1)}u_{i+j-2} - \tfrac{1}{6}(-1)^{c(j)}z_{i+j-1},$$
$$y_i * y_j = \tfrac{1}{6}(-1)^{c(j-2)}u_{i+j-1} + \tfrac{1}{6}(-1)^{c(j-1)}z_{i+j},$$

where u_i, z_i are assumed to be zero for $i \in \{4k-2, 4k-1\}$.

Furthermore, define on U a $\mathbb{Z}_2$-grading by putting

$$U_0 = \mathrm{vect}\langle x_{2k},\ y_{2k+1},\ u_{4k-3},\ z_{4k} \mid k > 0\rangle,$$
$$U_1 = \mathrm{vect}\langle x_{2k-1},\ y_{2k},\ u_{4k},\ z_{4k+1} \mid k > 0\rangle,$$

where $\mathrm{vect}\langle V\rangle$ denotes the vector space spanned by the set V.

Proposition 4.1. *The superalgebra $(U, *)$ is an alternative superalgebra generated by the odd element $x = x_1$.*

Proof. It is clear that

$$\begin{aligned}
x_{i+1} &= x_i * x - (-1)^i x * x_i, \quad i > 0,\\
y_i &= x_i * x, \quad i > 1,\\
u_1 &= -2x_2 * x_2,\\
u_{i+1} &= 6y_i * y_2 - 3x_{i+1} * y_2, \quad i+1 = 4k+\varepsilon,\\
z_{i+1} &= 3x_i * y_2, \quad i+1 = 4k+\varepsilon,
\end{aligned}$$

where $k > 0$, $\varepsilon \in \{0, 1\}$, hence U is generated by x.

Now we have to verify in U the identities of left and right super-alternativity. Consider first elements a, b from U^2. The product $a * b$ can be nonzero only when both a and b are of types x_i or y_i, $i > 1$. But in this case the result $a * b$ is a sum of the elements u_i and z_i. Hence $(U^2, U^2, U^2) = 0$. Now, if at least one argument is x, direct calculations yield for $i, j > 1$

$$\begin{aligned}
(x_i, x, x) &= (-1)^{i+1}(x, x_i, x) = (x, x, x_i)\\
&= \tfrac{1}{3}x_{i+2} - \tfrac{1}{6}z_i,\\
(y_i, x, x) &= (-1)^i(x, y_i, x) = (x, x, y_i)\\
&= \tfrac{1}{3}y_{i+2} + \tfrac{1}{4}u_i - \tfrac{1}{12}z_{i+1},\\
-(x_i, x, x_j) &= (-1)^i(x, x_i, x_j) = (-1)^j(x_i, x_j, x)\\
&= \tfrac{1}{3}(-1)^{c(j)}z_{i+j-1},\\
(y_i, x, y_j) &= (-1)^i(x, y_i, y_j) = (-1)^j(y_i, y_j, x)\\
&= \tfrac{1}{6}(-1)^{c(j)}u_{i+j} - \tfrac{1}{18}(-1)^{c(j+1)}z_{i+j+1},
\end{aligned}$$

$$\begin{aligned}(x_i, x, y_j) &= (-1)^{i+1}(x, x_i, y_j) = (-1)^j (x_i, y_j, x)\\ &= \tfrac{1}{6}(-1)^{c(j)}u_{i+j-1} - \tfrac{1}{6}(-1)^{c(j+1)}z_{i+j},\\ (y_j, x, x_i) &= (-1)^j (x, y_j, x_i) = (-1)^{i+1}(y_j, x_i, x)\\ &= (-1)^{i+1}\bigl(\tfrac{1}{6}(-1)^{c(i)}u_{i+j-1} + \tfrac{1}{6}(-1)^{c(i+1)}z_{i+j}\bigr),\end{aligned}$$

which easily implies that the super-alternative identities hold when one of the variables being permuted equals x. For the remaining cases we apply (3). For example,

$$\begin{aligned}(x, x_i, x_j) + (-1)^{ij}(x, x_j, x_i) &= \tfrac{1}{3}(-1)^{i+c(j)}z_{i+j-1} + \tfrac{1}{3}(-1)^{ij}(-1)^{j+c(i)}z_{i+j-1}\\ &= \tfrac{1}{3}(-1)^{i+c(j)}\bigl(1 + (-1)^{i+j+c(i)+c(j)+ij}\bigr)z_{i+j-1}\\ &\overset{(3)}{=} \tfrac{1}{3}(-1)^{i+c(j)}\bigl(1 + (-1)^{i+j+c(i+j)}\bigr)z_{i+j-1}\\ &\overset{(3)}{=} \tfrac{1}{3}(-1)^{i+c(j)}\bigl(1 + (-1)^{c(i+j+1)}\bigr)z_{i+j-1} = 0,\end{aligned}$$

since either $z_{i+j-1} = 0$ or $c(i + j + 1)$ is odd. □

Theorem 4.2. *The elements given in* (10) *form a base of the superalgebra* $\mathcal{E}$.

Proof. In fact, the superalgebra U is a homomorphic image of $\mathcal{E}$ under the homomorphism $x \mapsto x_1$. The images of the elements (10) form a base of U, hence these elements are linearly independent and form a base of $\mathcal{E}$. □

Theorem 4.3. *The free Malcev superalgebra $\mathcal{M}$ on one odd generator is special.*

Proof. The homomorphism $\varphi : \mathcal{M} \xrightarrow{\tau} \mathcal{A}^- \to \mathcal{E}^-$ maps the base

$$\mathcal{B}_{\mathcal{M}} = \bigl\{x^k,\ x^{4k}x^2,\ x^{4k+1}x^2 \mid k > 0\bigr\},$$

of $\mathcal{M}$ to the linearly independent elements $x^{[k]}$, $z^{[4k]}$, $z^{[4k+1]}$ in $\mathcal{E}$, consequently $\ker\varphi = 0$ and $\mathcal{M} \cong \varphi(\mathcal{M}) \subseteq \mathcal{E}^-$. □

The next corollaries follow easily from Theorem 4.3 in view of [7,9,13].

Corollary 4.4. *There are no nontrivial skew-symmetric Malcev s-identities.*

Corollary 4.5. *The Malcev Grassmann algebra* (*see* [9]) *is special.*

Corollary 4.6. *The elements* $\mathrm{Skew}\, z^{[4(k+1)]}$, $\mathrm{Skew}\, z^{[4k+1]}$, $k > 0$ (*see* [9,13]), *are nonzero central skew-symmetric elements in the free alternative algebra of countable rank.*

5. Speciality of homomorphic images

In this section we prove that every homomorphic image of the superalgebra $\mathcal{M}$ is special. We will identify $\mathcal{M}$ with its image in $U^- \cong \mathcal{E}^-$ and thus assume that $\mathcal{M} \subseteq U$. For an ideal K of $\mathcal{M}$ we denote by $\hat{K} = \mathrm{idl}_U\langle K\rangle$ the ideal of U generated by K.

Lemma 5.1. *Let K be an ideal of $\mathcal{M}$ such that $\hat{K} \cap \mathcal{M} = K$. Then the quotient superalgebra $\mathcal{M}/K$ is special.*

Proof. (see [12, Lemma 3.4]) Note first that for any ideal B of U there exists an isomorphism

$$(U/B)^- \cong U^-/B^- : (u+B)^- \mapsto u + B^-.$$

Now by the Second Isomorphism Theorem,

$$\mathcal{M}/K = \mathcal{M}/\big(\mathcal{M} \cap \hat{K}^-\big) \cong \big(\mathcal{M} + \hat{K}^-\big)/\hat{K}^- \subseteq U^-/\hat{K}^-,$$

which in view of the previous observation proves the speciality of $\mathcal{M}/K$. $\square$

Denote by Z the subsuperspace of $\mathcal{M}$ spanned by the elements z_i.

Lemma 5.2. *Let K, K_1, K_2 be ideals in $\mathcal{M}$ such that $K = K_1 + K_2$, $K_2 \subseteq Z$. If $\hat{K}_1 \cap \mathcal{M} = K_1$, then $\mathcal{M}/K$ is special.*

Proof. Note that $\hat{K} = \hat{K}_1 + \hat{K}_2$. Since $Z * U = U * Z = 0$ we have $\hat{K}_2 = K_2$. Now, since $\hat{K}_2 = K_2 \subseteq \mathcal{M}$ we have $\mathcal{M} \cap \hat{K} = \mathcal{M} \cap \hat{K}_1 + \hat{K}_2 = K_1 + K_2 = K$ and hence $\mathcal{M}/K$ is special by the previous lemma. $\square$

Lemma 5.3. *Let K be a proper ideal of $\mathcal{M}$ such that $K \nsubseteq Z$. Then there exist an element*

$$f = x_n + \sum_{1<i<n} \alpha_i x_i + z, \quad z \in Z, \tag{11}$$

and a finite set of elements $w_i \in Z$, $i \in \{1, \ldots, k\}$, such that

$$K = \mathrm{idl}_{\mathcal{M}}\langle f, w_1, \ldots, w_k\rangle.$$

Proof. Every element of $\mathcal{M}$ that does not belong to Z has the form

$$\sum_{0<i\leqslant n} \alpha_i x_i + z, \quad z \in Z,$$

for some n, α_i, z, and $\alpha_n \neq 0$. Choose an element f in K of this type with the minimal value of n and the leading coefficient $\alpha_n = 1$.

We show first that if $\alpha_1 \neq 0$ then $K = \mathcal{M}$. In fact, in this case the ideal K contains an element $x_2 f = \alpha_1 x_3 + z'$, $z' \in Z$, which implies $n < 3$. If $n = 1$ then $f = x + z$, $((fx)x\dots)x = x_i \in K$ and $z_i = x_i x_2 \in K$ for all $i > 1$. Therefore, $z \in K$ and $x \in K$, yielding $K = \mathcal{M}$. If $n = 2$ then $f = x_2 + \alpha x + z$, $\alpha \neq 0$. We have $x_2 f = \alpha x_3 \in K$, which yields $x_i \in K$ for all $i > 2$. Now $fx = x_3 + \alpha x_2 \in K$, hence $x_2 \in K$ and $n < 2$, a contradiction.

Thus we may assume that the element f has a form (11). Show that f satisfies the conclusion of the lemma. Let g be an arbitrary element in $K \setminus Z$, say

$$g = x_m + \sum_{i<m} \beta_i x_i + z', \quad z' \in Z.$$

We prove by induction on m that $g \in \mathrm{idl}_{\mathcal{M}}\langle f\rangle + Z$. If $m = n$ then by minimality of n we have $\alpha_i = \beta_i$ and $f - g = z - z' \in Z$. Suppose now that $m > n$. Write

$$f_k = \big((f\underbrace{x)x\dots\big)x}_{k} = x_{n+k} + \sum_{1<i<n} \alpha_i x_{i+k}, \quad k > 0. \tag{12}$$

The difference $g - f_{m-n}$ belongs to K and modulo Z is a linear combination of x_i, $i < m$. By the induction assumption, $g - f_{m-n} \in \mathrm{idl}_{\mathcal{M}}\langle f\rangle + Z$, hence $K \subseteq \mathrm{idl}_{\mathcal{M}}\langle f\rangle + Z$.

Furthermore, the ideal $\mathrm{idl}_{\mathcal{M}}\langle f\rangle$ contains the elements

$$z_{n+k} + \sum_{1<i<n} \alpha_i z_{i+k}, \quad k \geqslant 0. \tag{13}$$

Hence we can express all z_k, $k \geqslant n$, modulo $\mathrm{idl}_{\mathcal{M}}\langle f\rangle$ as linear combinations of the z_i, $i < n$. Since the space $\mathrm{vect}\langle z_4, \dots, z_{n-1}\rangle$ is finite dimensional, this implies the conclusion of the lemma. □

Theorem 5.4. *For any ideal K, the quotient superalgebra $\mathcal{M}/K$ is special.*

Proof. By the two previous lemmas, it suffices to prove that $\hat{K} \cap \mathcal{M} = K$, where $K = \mathrm{idl}_{\mathcal{M}}\langle f\rangle$ and f is an element of form (11). We show that the ideal $\hat{K}$ coincides with the subspace V of U spanned by the elements

$$f, \qquad x_{n+m+1} + \sum_{1<i<n} \alpha_i x_{i+m+1}, \qquad y_{n+m} + \sum_{1<i<n} \alpha_i y_{i+m},$$
$$z_{n+m} + \sum_{1<i<n} \alpha_i z_{i+m}, \qquad u_{n+m-1} + \sum_{1<i<n} \alpha_i u_{i+m-1}, \quad m \geqslant 0.$$

In fact, all these elements lie in $\hat{K} = \mathrm{idl}_U\langle f\rangle$. On the other hand, it is easy to see that V is closed with respect to multiplication by the elements of the base of U and hence is an ideal of U. Since $f \in V$, we have $V = \hat{K}$. Evidently, the subspace $V \cap \mathcal{M}$ is spanned by the elements

$$f, \qquad x_{n+m+1} + \sum_{1<i<n} \alpha_i x_{i+m+1}, \qquad z_{n+m} + \sum_{1<i<n} \alpha_i z_{i+m}, \quad m \geqslant 0.$$

By (12), (13), all these elements lie in K, hence $\hat{K} \cap \mathcal{M} \subseteq K$. Since the inverse inclusion is evident, we have $\hat{K} \cap \mathcal{M} = K$, proving the theorem. □

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